The Selberg zeta function and a new Kähler metric on the moduli space of punctured Riemann surfaces

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Dedicated to I.M. Gelfand
on his 75th birthday

Abstract. *A local index theorem for families of $\bar\partial$-operators on Riemann surfaces with punctures is proved. A new Kähler metric on the moduli space of punctured surfaces is described in terms of the Eisenstein - Maass series.*

0. The well-known papers by Quillen [1] and Belavin - Knizhnik [2] first revealed an important role of functional determinants in the context of the Atiyah - Singer index theorem for families. Namely, for families of $\bar\partial$-operators on compact Riemann surfaces they evaluated explicitly the curvature form of the so-called Quillen's metric in the determinant line bundle, obtained by multiplying the ordinary L^2-metric by det $\bar\partial^*\bar\partial$ (the determinant of the corresponding Laplace operator). In the simplest case of $\bar\partial$-operators acting on functions one has the following formula:

$$(1) \qquad \partial\bar\partial \log \frac{\det \Delta}{\det \operatorname{Im} \tau} = \frac{\sqrt{-1}}{6\pi}\, \omega_{WP}.$$

Here Δ is the Laplace operator corresponding to the Poincaré metric on a

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compact Riemann surface X, τ is the period matrix for X (both det Δ and det Im τ are considered locally as functions on the moduli space $\mathcal{U}_g$ of compact Riemann surfaces of genus $g \geqslant 2$), ∂ and $\bar{\partial}$ are the components of the exterior derivative operator on $\mathcal{U}_g$, and, finally, ω_{WP} is the symplectic form of the Weil - Petersson metric on $\mathcal{U}_g$. An "elementary" proof of this formula one can find in [3].

In this paper we tried to extend the ideas of [1, 2] to the case of noncompact Riemann surfaces as well. Unfortunately, the Laplace operator has then a continuous spectrum, and there is no conventional definition of determinant in this situation. Here, as in [3], by det Δ we understand the value $Z'(1)$, where $Z(s)$ is the Selberg zeta function for the corresponding Riemann surface (it is known that for compact Riemann surfaces $Z'(1)$ coincides, up to a constant depending only on g, with detΔ defined, as usual, by means of the operator zeta function). Using an approach similar to [3, 4] we are able to derive an analog of formula (1) for the moduli space of Riemann surfaces with punctures, which differs from (1) by an additional term; see formula (12) at the end of the paper. This new term is the symplectic form of a Kähler metric on the moduli space defined by means of the Eisenstein - Maass series. We also show that this Kähler metric does not in general coincide with the Weil - Petersson metric. In the language of physics one can say that punctures give an additional contribution to the so-called "holomorphic anomaly" (cf. [2]).

With a great pleasure and sincere feelings we dedicate this paper to I. M. Gelfand on the occasion of his 75th birthday. We hope that the topics of our paper lie within the scope of his mathematical interests and remind him of the good old days of "Representation theory and automorphic functions" [5].

1. First we will remind basic facts about automorphic functions and spectral properties of the Laplace operator on noncompact Riemann surfaces; for further details and proofs, see, e.g., [5-7].

Let X be a Riemann surface of type (g, n), i.e. $X = \bar{X} \setminus \{x_1, \dots, x_n\}$, where $\bar{X}$ is a compact Riemann surface of genus g and $x_1, \dots, x_n$ are pairwise distinct points on $\bar{X}$. In addition we will assume that $2g + n \geqslant 3$. In this case X can be represented as a quotient $\Gamma \setminus H$ of the upper half-plane $H = \{z = x + \sqrt{-1}\, y \in \mathbb{C} \mid y > 0\}$ by the action of a torsion-free finitely generated Fuchsian group Γ. The group $\Gamma \subset PSL(2, \mathbb{R})$ is generated by $2g$ hyperbolic transformations $A_1, B_1, \dots, A_g, B_g$ and n parabolic transformations $S_1, \dots, S_n$ satisfying the single relation $A_1 B_1 A_1^{-1} B_1^{-1} \dots A_g B_g A_g^{-1} B_g^{-1} S_1 \dots S_n = 1$. The fixed points of the parabolic elements $S_1, \dots, S_n$ (cusps) will be denoted by $z_1, \dots, z_n$ respectively. The "images" of the cusps $z_1, \dots, z_n \in \mathbb{R} \cup \{\infty\}$ under the projection map $H \to \Gamma \setminus H \cong X$ are the punctures $x_1, \ldots, x_n \in \bar{X}$. For each $i = 1, \ldots, n$ denote by Γ_i the cyclic subgroup in Γ generated by S_i and choose an element $\delta_i \in PSL(2, \mathbb{R})$ such that

$$\delta_i \infty = z_i \text{ and } \delta_i^{-1} S_i \delta_i = \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}$$

A smooth complex valued function f on H is called an automorphic form of weight $2l$, $l \in \mathbb{Z}$, with respect to the group Γ if $f(\gamma z)\, \gamma'(z)^l = f(z)$ for all $\gamma \in \Gamma$ and $z \in H$ (forms of weight $2l$ correspond to l-differentials on the Riemann surface $X \cong \Gamma \setminus H$). A holomorphic automorphic form f is called regular if at each cusp z_i it has the following Fourier expansion:

$$f(\delta_i z)\, \delta_i(z)^l = \sum_{k=0}^{\infty} a_k^{(i)}\, e^{2\pi \sqrt{-1} k z}, \qquad i = 1, \ldots, n.$$

If, moreover, $a_0^{(1)} = \ldots = a_0^{(n)} = 0$, f is called a cusp form. Denote by $\Omega_l(\Gamma)$ the linear space of cusp forms of weight $2l$ for the group Γ; by Riemann-Roch

$$\dim \Omega_l(\Gamma) = \begin{cases} 0, & l \leqslant 0, \\ g, & l = 1, \\ (2l-1)g + (l-1)n, & l \geqslant 2. \end{cases}$$

Now we will turn to the spectral properties of the Laplace operator on the Riemann surface $X \cong \Gamma \setminus H$. Let $\rho(z)\,|\,dz\,|^2 = y^{-2}(dx^2 + dy^2)$ be the Poincarè metric on H. Denote by $\mathscr{H}(\Gamma)$ the Hilbert space of measurable complex valued functions on $X \cong \Gamma \setminus H$ with respect to the scalar product

$$\langle f_1, f_2 \rangle = \int_X f_1 \bar{f}_2\, \rho =$$

$$= \int_{\Gamma \setminus H} f_1(z) \overline{f_2(z)}\, \frac{dx\, dy}{y^2}, \qquad f_1, f_2 \in \mathscr{H}(\Gamma).$$

The corresponding Laplace operator

$$\Delta = -4y^2 \frac{\partial^2}{\partial z\, \partial \bar{z}} = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

is self-adjoint and non-negative in the space $\mathscr{H}(\Gamma)$. Denote by G_s the resolvent

$(\Delta + s(s-1))^{-1}$ of the Laplace operator Δ in $\mathscr{H}(\Gamma)$. For $\operatorname{Re} s > 1$ and $z \neq \gamma z'$, $\gamma \in \Gamma$, its kernel $G_s(z, z')$ is given by the absolutely convergent series

$$G_s(z, z') = \sum_{\gamma \in \Gamma} Q_s(z, \gamma z'),$$

where $Q_s(z, z')$ is the resolvent kernel of the Laplace operator Δ on the upper half-plane H. The kernel $Q_s(z, z')$ is smooth for $z \neq z'$ and is holomorphic in s on the whole complex s-plane. It also has an important property that $Q_s(\delta z, \delta z') = Q_s(z, z')$ for any $\delta \in PSL(2, \mathbb{R})$ and $z, z' \in H$. At $s = 1$ one has

$$Q_1(z, z') = -\frac{1}{2\pi} \log \left| \frac{z - z'}{z - \bar{z}'} \right| .$$

The kernel $G_s(z, z')$ with $z \neq \gamma z'$, $\gamma \in \Gamma$, admits a meromorphic continuation in s to the entire complex plane and has the following Laurent expansion at $s = 1$:

$$G_s(z, z') = \frac{1}{4\pi\left(g - 1 + \frac{n}{2}\right)} \cdot \frac{1}{s(s-1)} + G_1^0(z, z') + \underset{s \to 1}{O(s-1)}$$

(see [7, Th. 2.3]). The kernel $G_1^0(z, z')$ is called the Green function of the Laplace operator Δ on the Riemann surface $X \cong \Gamma \backslash H$.

Side by side with functions, we will also consider tensors of type (l, m) on Riemann surfaces. On the upper half-plane H they are represented by automorphic forms of weight $(2l, 2m)$, i.e. by functions satisfying the following transformation law:

$$f(\gamma z)\,(\gamma'(z))^l\,\overline{\gamma'(z)}^m = f(z), \qquad \gamma \in \Gamma,\ z \in H.$$

By $\mathscr{H}_{l,m}(\Gamma)$ we will denote the Hilbert space of such forms with respect to the scalar product

$$(2) \qquad \langle f_1, f_2 \rangle = \int_{\Gamma \backslash H} f_1(z)\,\overline{f_2(z)}\,y^{2l+2m-2}\,dx\,dy, \quad f_1, f_2 \in \mathscr{H}_{l,m}(\Gamma).$$

Elements of the space $\mathscr{H}_{-1,1}(\Gamma)$ are usually called Beltrami differentials.

2. Now let us proceed with necessary facts from the theory of Teichmüller spaces. Let $T_{g,n}$ be the Teichmüller space of marked Riemann surfaces of genus g with n punctures (we identify it with the Teichmüller space of the marked

Fuchsian group Γ uniformizing the Riemann surface X). The Teichmüller space $T_{g,n}$ admits a natural complex manifold structure of dimension $3g - 3 + n$. For its description consider in the Hilbert space $\mathcal{H}_{-1,1}(\Gamma)$ the subspace $\Omega_{-1,1}(\Gamma)$ of harmonic Beltrami differentials; each element $\mu \in \Omega_{-1,1}(\Gamma)$ has a form $\mu = y^2\overline{\varphi}$, $\varphi \in \Omega_2(\Gamma)$, so $\dim \Omega_{-1,1}(\Gamma) = 3g - 3 + n$. The space $\Omega_{-1,1}(\Gamma)$ is naturally isomorphic to the tangent space $T_{[X]}\, T_{g,n}$ to the Teichmüller space $T_{g,n}$ at the point $[X]$ representing the (marked) Riemann surface X. In turn, the cotangent space $T^*_{[X]}\, T_{g,n}$ can be identified with the space $\Omega_2(\Gamma)$, which is dual to $\Omega_{-1,1}(\Gamma)$ with respect to the pairing

$$(\mu, \varphi) = \int_X \mu\varphi, \qquad \mu \in \Omega_{-1,1}(\Gamma), \quad \varphi \in \Omega_2(\Gamma).$$

For every $\mu \in \Omega_{-1,1}(\Gamma)$ with

$$\|\mu\|_\infty = \sup_{z \in H} |\mu(z)| < 1$$

there exists a unique diffeomorphism $f^\mu : H \to H$ satisfying the Beltrami equation

$$\frac{\partial f^\mu}{\partial \bar{z}} = \mu \frac{\partial f^\mu}{\partial z}$$

and fixing the points $0, 1, \infty$.

Set $\Gamma^\mu = f^\mu \Gamma (f^\mu)^{-1}$ and $X^\mu = \Gamma^\mu \backslash H$. Choose a basis $\mu_1, \ldots, \mu_{3g-3+n}$ in the linear space $\Omega_{-1,1}(\Gamma)$ and let $\mu = \epsilon_1 \mu_1 + \ldots + \epsilon_{3g-3+n}\mu_{3g-3+n}$. Then the correspondence $(\epsilon_1, \ldots, \epsilon_{3g-3+n}) \mapsto [X^\mu]$ defines complex coordinates in a neighbourhood of the point $[X] \in T_{g,n}$. They are called the Bers coordinates. In the overlapping neighbourhoods of two points $[X]$ and $[X^\mu]$ the Bers coordinates transform complex analytically. The differential of this coordinate change at the point $[X] \in T_{g,n}$ is a linear map $D_\mu : \Omega_{-1,1}(\Gamma) \to \Omega_{-1,1}(\Gamma^\mu)$. With the Bers coordinates $(\epsilon_1, \ldots \epsilon_{3g-3+n})$ in a neighbourhood of the point $[X] \in T_{g,n}$ one can associate $3g - 3 + n$ vector fields $\partial/\partial\epsilon_i$. At any other point $[X^\mu]$, $\mu \in \Omega_{-1,1}(\Gamma)$, in this neighbourhood they are represented by the Beltrami differentials $D_\mu \mu_i \in \Omega_{-1,1}(\Gamma^\mu)$, $i = 1, \ldots, 3g - 3 + n$. Further details can be found in [8, 9].

Due to the isomorphism $T_{[X]}\, T_{g,n} \cong \Omega_{-1,1}(\Gamma)$, the scalar product (2) defines a Hermitian metric on the Teichmüller space $T_{g,n}$, which is called the Weil-Petersson metric. This metric is Kähler [8], and its symplectic form will be denoted by ω_{WP};

$$\omega_{WP}(\mu, \bar{\nu}) = \frac{\sqrt{-1}}{2} \langle \mu, \nu \rangle.$$

3. Here we will collect necessary variational formulas connected with Teichmüller spaces. First consider a smooth family $\omega^\epsilon \in \mathscr{H}_{l,m}(\Gamma^{\epsilon\mu})$ of tensors of type (l, m), where $\mu \in \Omega_{-1,1}(\Gamma)$ and $\epsilon \in \mathbb{C}$ is sufficiently small. Set

$$f_*^{\epsilon\mu}(\omega^\epsilon) = \omega^\epsilon \circ f^{\epsilon\mu} \left(\frac{\partial f^{\epsilon\mu}}{\partial z}\right)^l \left(\overline{\frac{\partial f^{\epsilon\mu}}{\partial z}}\right)^m \in \mathscr{H}_{l,m}(\Gamma).$$

The Lie derivative of the family ω^ϵ in the tangental direction $\mu \in \Omega_{-1,1}(\Gamma)$ is

$$L_\mu \omega = \frac{\partial}{\partial \epsilon}\bigg|_{\epsilon=0} f_*^{\epsilon\mu}(\omega^\epsilon) \in \mathscr{H}_{l,m}(\Gamma);$$

similarly,

$$L_{\bar{\mu}} \omega = \frac{\partial}{\partial \bar{\epsilon}}\bigg|_{\bar{\epsilon}=0} f_*^{\epsilon\mu}(\omega^\epsilon) \in \mathscr{H}_{l,m}(\Gamma).$$

For the density of the Poincaré metric $\rho(z) = y^{-2}$, regarded as a family of tensors of type $(1, 1)$, Ahlfors proved in [8] the following formula:

(3) $$L_\mu \rho = L_{\bar{\mu}} \rho = 0.$$

For another important family $\mu^{\epsilon\nu} = D_{\epsilon\nu}\mu \in \Omega_{-1,1}(\Gamma^{\epsilon\nu})$, where $\mu, \nu \in \Omega_{-1,1}(\Gamma)$, one has

(4) $$L_{\bar{\nu}} \mu = -4 \frac{\partial}{\partial \bar{z}} \left(y^2 \frac{\partial}{\partial \bar{z}} (\Delta + 2)^{-1} (\mu\bar{\nu}) \right)$$

(see [9, Th. 2.9].)

Later we will need the following

LEMMA 1. *Set* $f_{\mu\bar{\nu}} = (\Delta + 2)^{-1}(\mu\bar{\nu}) \in \mathscr{H}(\Gamma)$, *where* $\mu, \nu \in \Omega_{-1,1}(\Gamma)$. *Then near each cusp* z_i, $i = 1, \ldots, n$, *of the group* Γ

$$f_{\mu\bar{\nu}}(\delta_i z) = \frac{c_{\mu\bar{\nu}}^{(i)}}{y} + \textit{ exponentially decreasing terms as } y \to \infty,$$

where $c_{\mu\bar{\nu}}^{(i)}$ *is a constant not depending on* $z \in H$.

Proof. Since $\mu, \nu \in \Omega_{-1,1}(\Gamma)$, one has $\mu = y^2\bar{\varphi}$, $\nu = y^2\bar{\psi}$ for some cusp forms $\varphi, \psi \in \Omega_2(\Gamma)$, and hence the function $\mu\bar{\nu} \in \mathscr{H}(\Gamma)$ is exponentially decreasing at the cusps $z_1, \ldots, z_n$. Let

$$f_{\mu\bar{\nu}}(\delta_i z) = \sum_{k=-\infty}^{\infty} a_k^{(i)}(y)\, e^{2\pi\sqrt{-1}kx}$$

be the Fourier expansion of the function $f_{\mu\bar{\nu}}$ at the cusp z_i, $i = 1, \ldots, n$. Because $(\Delta + 2) f_{\mu\bar{\nu}} = \mu\bar{\nu}$, each function

$$\frac{d^2 a_k^{(i)}}{dy^2} + \left(4\pi^2 k^2 - \frac{2}{y^2}\right) a_k^{(i)}$$

is exponentially decreasing as $y \to \infty$. The equation

$$\frac{d^2 a}{dy^2} + \left(4\pi^2 k^2 - \frac{2}{y^2}\right) a = 0$$

has a pair of linearly independent solutions $1/y$, y^2 when $k = 0$, and

$$\sqrt{y}\, K_{3/2}(2\pi|k|y) \underset{y\to\infty}{\sim} e^{-2\pi|k|y},\ \sqrt{y}\, I_{3/2}(2\pi|k|y) \underset{y\to\infty}{\sim} e^{2\pi|k|y}$$

when $k \neq 0$. Since $f_{\mu\bar{\nu}} \in \mathscr{H}(\Gamma)$, increasing solutions cannot occur in the Fourier expansion of $f_{\mu\bar{\nu}}$, and we immediately arrive at the assertion of the lemma. ∎

The Lie derivative of a family of linear operators $A^\epsilon : \mathscr{H}_{l,m}(\Gamma^{\epsilon\mu}) \to \mathscr{H}_{l',m'}(\Gamma^{\epsilon\mu})$ is defined by the formula

$$L_\mu A = \left.\frac{\partial}{\partial\epsilon}\right|_{\epsilon=0} (f_*^{\epsilon\mu} A^\epsilon (f_*^{\epsilon\mu})^{-1});$$

$L_\mu A$ is a linear operator from $\mathscr{H}_{l,m}(\Gamma)$ to $\mathscr{H}_{l',m'}(\Gamma)$. For the family of the Laplace operators Δ one can easily derive a formula

$$L_\mu \Delta = 4y^2 \frac{\partial}{\partial z}\mu \frac{\partial}{\partial z}, \tag{5}$$

where $\mu \in \Omega_{-1,1}(\Gamma)$ (see, e.g., [4]).

4. Now we will busy ourselves with abelian differentials and classical kernels on Riemann surfaces. Let $X \cong \Gamma \setminus H$ be a marked Riemann surface of type (g, n) and let $\omega_1, \ldots, \omega_g$ be the normalized basis of the linear space $\Omega_1(\Gamma)$, i.e.

$$\int_z^{A_i z} \omega_j(z)\, dz = \delta_{ij}, \qquad i, j = 1, \ldots, g$$

where δ_{ij} is the Kronecker symbol. The period matrix $\tau = (\tau_{ij})$ of X defined by

$$\tau_{ij} = \int_z^{B_i z} \omega_j(z)\, dz,$$

has a symmetric positive definite imaginary part $\operatorname{Im} \tau = (\operatorname{Im} \tau_{ij})$ with the property

$$\operatorname{Im} \tau_{ij} = \int_{\Gamma \backslash H} \omega_i(z)\, \overline{\omega_j(z)}\, dx\, dy, \qquad i, j = 1, \ldots, g.$$

Moreover, for every $\mu \in \Omega_{-1,1}(\Gamma)$ and $i = 1, \ldots, g$ one has $L_{\bar{\mu}} \omega_i = 0$ (see [10]). The first derivatives of the period matrix with respect to coordinates on the Teichmüller space $T_{g,n}$ are given by Rauch's formulas [11]:

$$L_\mu \tau_{ij} = -2\sqrt{-1} \int_{\Gamma \backslash H} \omega_i\, \omega_j\, \mu, \qquad L_{\bar{\mu}}\, \tau_{ij} = 0,$$

$$\mu \in \Omega_{-1,1}(\Gamma), \qquad i, j = 1, \ldots, g.$$

This immediately yields

$$L_\mu (\log \det \operatorname{Im} \tau) = \operatorname{tr}\, ((\operatorname{Im} \tau)^{-1} L_\mu (\operatorname{Im} \tau)) = \tag{6}$$

$$= -\int_{\Gamma \backslash H} \sum_{i,j=1}^{g} (\operatorname{Im} \tau)^{-1}_{ij}\, \omega_i\, \omega_j\, \mu.$$

In other words, the (1, 0)-form $\partial \log \det \operatorname{Im} \tau$ on the Teichmüller space $T_{g,n}$ corresponds under the isomorphism $T^*_{[X]} T_{g,n} \cong \Omega_2(\Gamma)$ to the family of cusp forms

$$-\sum_{i,j=1}^{g} (\operatorname{Im} \tau)^{-1}_{ij}\, \omega_i\, \omega_j.$$

For a marked Riemann surface $X \cong \Gamma \backslash H$ denote by $B(z, z')$ the uniquely determined symmetric bidifferential of the second kind on $\overline{X} \times \overline{X}$ with a double pole of biresidue 1 at the diagonal $z = z'$ and zero A-periods; its B-periods are given by

$$\int_{z}^{B_i z} B(z, z')\, dz = 2\pi \sqrt{-1}\, \omega_i(z'), \qquad i = 1, \ldots, g$$

(see [7, p. 160]). It is clear that $L_{\bar{\mu}} B = 0$ for any $\mu \in \Omega_{-1,1}(\Gamma)$ (because $L_{\bar{\mu}} B$ is a regular bidifferential on $\overline{X} \times \overline{X}$ with zero periods).

The Schiffer kernel $\Omega(z, z')$ is defined as a symmetric bidifferential of the second kind on $\overline{X} \times \overline{X}$ with a double pole of biresidue 1 at the diagonal $z = z'$ and the property

$$v.p. \int_{\Gamma \backslash H} \Omega(z, z')\, \overline{\omega(z')}\, dx'\, dy' = 0$$

for every $\omega \in \Omega_1(\Gamma)$. The Schiffer kernel does not depend on a marking of X. Moreover, the following formulas hold:

$$\Omega(z, z') = B(z, z') - \pi \sum_{i,j=1}^{g} (\operatorname{Im} \tau)^{-1}_{ij}\, \omega_i(z)\, \omega_j(z') \tag{7}$$

and

$$\Omega(z, z') = -4\pi \frac{\partial^2}{\partial z\, \partial z'} G_1^0(z, z') = -4\pi \lim_{s \to 1} \frac{\partial^2}{\partial z \partial z'} G_s(z, z') \tag{8}$$

(see [7, p. 161].

5. The Selberg zeta function $Z(s)$ of a Riemann surface X is defined for $\operatorname{Re} s > 1$ by the absolutely convergent product

$$Z(s) = \prod_{\{l\}} \prod_{k=0}^{\infty} (1 - e^{-(s+k)|l|}),$$

where l runs over the set of all simple closed geodesics on $X \cong \Gamma \backslash H$ supplied with the Poincaré metric, and $|\, l\,|$ is the length of a geodesic l. The function $Z(s)$ has a meromorphic continuation to the entire complex s-plane with a simple zero at $s = 1$. For the logarithmic derivative of $Z(s)$ one has

$$\frac{1}{2s-1} \frac{d}{ds} \log Z(s) = \tag{9}$$

$$= \int_{\Gamma\backslash H} \sum_{\substack{\gamma\in\Gamma \\ \gamma\ hyperbolic}} Q_s(z, \gamma z) \frac{dx\,dy}{y^2}, \tag{9}$$

where the sum is taken over all hyperbolic elements of the group Γ. The last formula can be easily deduced from the definition of $Z(s)$ with the help of the Selberg transform. Further details concerning the Selberg zeta function can be found in [12].

THEOREM 1. *For every* $\mu \in \Omega_{-1,1}(\Gamma)$

$$L_\mu \log Z'(1) = -4 \int_{\Gamma\backslash H} R\,|_D \cdot \mu$$

where

$$R(z, z') = \frac{\partial^2}{\partial z\, \partial z'} (G_1^0(z, z') - Q_1(z, z'))$$

and D *denotes the diagonal* $z = z'$ *in* $H \times H$.

Proof. As it follows from the properties of the Green function $G_1^0(z, z')$ (see [7, Th. 2.3]), $R\,|_D$ is a regular holomorphic automorphic form of weight 4 for the group Γ, and the integral in the right hand side of the above formula is convergent because of $\mu \in \Omega_{-1,1}(\Gamma)$.

Further, since $Z(s)$ has a simple zero at $s = 1$, we have

$$L_\mu \log Z'(1) = \lim_{s\to 1} L_\mu \log Z(s).$$

Differentiating now both sides of formula (9) and taking (3) into account, we obtain that for $\operatorname{Re} s > 1$

$$\frac{1}{2s-1} L_\mu \left(\frac{d}{ds} \log Z(s)\right) = \int_{\Gamma\backslash H} L_\mu \Bigg(\sum_{\substack{\gamma\in\Gamma \\ \gamma\ hyperbolic}} Q_s(z, \gamma z)\Bigg) \frac{dx\,dy}{y^2} \tag{10}$$

$$= \int_{\Gamma\backslash H} \Bigg(L_\mu G_s(z, z') - L_\mu Q_s(z, z') - \sum_{\substack{\gamma\in\Gamma \\ \gamma\ parabolic}} L_\mu Q_s(z, \gamma z')\Bigg)\Bigg|_{z'=z}$$

$$\frac{dx\,dy}{y^2},$$

where

$$L_\mu Q_s(z, z') = \frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} Q_s(f^{\epsilon\mu}(z), f^{\epsilon\mu}(z'))$$

and

$$L_\mu G_s(z, z') = \frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} G_s^{\epsilon\mu}(f^{\epsilon\mu}(z), f^{\epsilon\mu}(z'))$$

($G_s^{\epsilon\mu}$ stands for the resolvent of the Laplace operator on the Riemann surface $X^{\epsilon\mu} \cong \Gamma^{\epsilon\mu} \setminus H$). Denote by $G_s^{(i)}(z, z')$ the resolvent kernel of the Laplace operator on the Riemann surface $\Gamma_i \setminus H$; for $\operatorname{Re} s > 1$

$$G_s^{(i)}(z, z') = \sum_{\gamma \in \Gamma_i} Q_s(z, \gamma z'), \qquad i = 1, \ldots, n.$$

From the definition of the resolvent and formula (5) for $L_\mu \Delta$ it follows that

$$L_\mu Q_s(z, z') = -\, v.p. \int_H Q_s(z, z'')(L_\mu \Delta)'' Q_s(z'', z') \frac{dx''\, dy''}{y''^2} =$$

$$= 4 \int_H \mu(z'') \frac{\partial}{\partial z''} Q_s(z, z'') \frac{\partial}{\partial z''} Q(z'', z')\, dx''\, dy'', \quad z \neq z',$$

$$L_\mu G_s^{(i)}(z, z') = -\, v.p. \int_{\Gamma_i \setminus H} G_s^{(i)}(z, z'')\, (L_\mu \Delta)''\, G_s^{(i)}(z'', z') \frac{dx''dy''}{y''^2} =$$

$$= 4 \int_{\Gamma_i \setminus H} \mu(z'') \frac{\partial}{\partial z''} G_s^{(i)}(z, z'') \frac{\partial}{\partial z''} G_s^{(i)}(z'', z')\, dx''\, dy'',$$

$$z \neq \gamma z', \ \gamma \in \Gamma_i,$$

and

$$L_\mu G_s(z, z') = -\, v.p. \int_{\Gamma \setminus H} G_s(z, z'')(L_\mu \Delta)''\, G_s(z'', z') \frac{dx''dy''}{y''^2} =$$

$$= 4 \int_{\Gamma \setminus H} \mu(z'') \frac{\partial}{\partial z''} G_s(z, z'') \frac{\partial}{\partial z''} G_s(z'', z')\, dx''\, dy'',$$

$$z \neq \gamma z', \ \gamma \in \Gamma,$$

where $(L_\mu \Delta)''$ means that the differential operator $L_\mu \Delta$ acts in the variable z''. Using now the above expressions for $L_\mu Q_s$, $L_\mu G_s^{(i)}$, $L_\mu G_s$ and a simple formula

$$\sum_{\substack{\gamma\in\Gamma \\ \gamma \text{ parabolic}}} Q_s(z, \gamma z') =$$

$$= \sum_{i=1}^{n} \sum_{\delta \in \Gamma_i \backslash \Gamma} (G_s^{(i)}(\delta z, \delta z') - Q_s(z, z')),$$

we derive from (10) the following formula:

$$\frac{1}{2s-1} L_\mu \left(\frac{d}{ds} \log Z(s) \right) =$$

$$= 4 \int_{\Gamma\backslash H} \mu(z)\, dx\, dy \left. \frac{\partial^2}{\partial z\, \partial z'} \right|_{z'=z} \left(\int_{\Gamma\backslash H} G_s(z, z'')\, G_s(z'', z') \right.$$

$$\frac{dx''\, dy''}{y''^2} - \int_H Q_s(z, z'')\, Q_s(z'', z') \frac{dx''dy''}{y''^2} -$$

$$- \sum_{i=1}^{n} \sum_{\delta \in \Gamma_i\backslash\Gamma} \left(\int_{\Gamma_i\backslash H} G_s^{(i)}(\delta z, \delta z'')\, G_s^{(i)}(\delta z'', \delta z') \frac{dx''\, dy''}{y''^2} - \right.$$

$$\left. \left. - \int_H Q_s(z, z'')\, Q_s(z'', z') \frac{dx''\, dy''}{y''^2} \right) \right) =$$

$$= \frac{4}{1-2s} \int_{\Gamma\backslash H} \mu(z)\, dx\, dy \left. \frac{\partial^2}{\partial z\, \partial z'} \right|_{z'=z} \frac{d}{ds} \Big(G_s(z, z') -$$

$$- Q_s(z, z') - \sum_{i=1}^{n} \sum_{\delta \in \Gamma_i\backslash\Gamma} (G_s^{(i)}(\delta z, \delta z') - Q_s(z, z')) \Big)$$

$$= \frac{4}{1-2s} \frac{d}{ds} \int_{\Gamma \backslash H} \frac{\partial^2}{\partial z \, \partial z'} (G_s(z, z') - Q_s(z, z') -$$

$$- \sum_{\substack{\gamma \in \Gamma \\ \gamma \text{ parabolic}}} Q_s(z, \gamma z'))\Big|_{z'=z} \mu(z) \, dx \, dy, \quad \operatorname{Re} s > 1$$

(here, after reversing the order of integration, we applied Hilbert's identity to the resolvent kernels). Integration of the last formula in s over the interval $[a, b] \subset \mathbb{R}$, $1 < a < b$, yields

$$L_\mu \log Z(s) \Big|_{s=a}^{s=b} =$$

$$= -4 \int_{\Gamma \backslash H} \frac{\partial^2}{\partial z \, \partial z'} \Bigg(G_s(z, z') - Q_s(z, z') -$$

$$- \sum_{\substack{\gamma \in \Gamma \\ \gamma \text{ parabolic}}} Q_s(z, \gamma z') \Bigg) \Bigg|_{z'=z} \mu(z) \, dx \, dy \Big|_{s=a}^{s=b} .$$

In the limit as $a \to 1$ and $b \to \infty$ we get

$$(11) \qquad L_\mu \log Z'(1) = -4 \int_{\Gamma \backslash H} \frac{\partial^2}{\partial z \, \partial z'} (G_1(z, z') - Q_1(z, z'))\big|_{z'=z}$$

$$\mu(z) \, dx \, dy + 4 \int_{\Gamma \backslash H} \sum_{\substack{\gamma \in \Gamma \\ \gamma \text{ parabolic}}} \frac{\partial^2}{\partial z \, \partial z'} Q_1(z, \gamma z') \Big|_{z'=z} \mu(z) \, dx \, dy,$$

because $L_\mu \log Z(s) \xrightarrow[s \to \infty]{} 0$ and

$$\sum_{\substack{\gamma \in \Gamma \\ \gamma \text{ hyperbolic}}} \frac{\partial^2}{\partial z \, \partial z'} Q_s(z, \gamma z') \Big|_{\substack{s \to \infty \\ z'=z}} \to 0$$

uniformly in $z \in H$.

In order to complete the proof it remains to show that the second integral in the right hand side of (11) vanishes for every $\mu \in \Omega_{-1,1}(\Gamma)$. We have

$$\sum_{\substack{\gamma \in \Gamma, \\ \gamma \text{ parabolic}}} \frac{\partial^2}{\partial z \, \partial z'} Q_1(z, \gamma z') \Big|_{z'=z} =$$

$$= 2 \sum_{i=1}^{n} \sum_{\delta \in \Gamma_i \backslash \Gamma} \sum_{k=1}^{\infty} \frac{\partial^2}{\partial z\, \partial z'} Q_1(z, \delta^{-1} S_i^k \delta z')\Big|_{z'=z} =$$

$$= 2 \sum_{i=1}^{n} \sum_{\delta \in \Gamma_i \backslash \Gamma} \sum_{k=1}^{\infty} \frac{\partial^2}{\partial z\, \partial z'} Q_1(\delta_i^{-1} \delta z, \delta_i^{-1} \delta z' + k)\Big|_{z'=z}$$

$$= -\frac{1}{2\pi} \sum_{i=1}^{n} \sum_{\delta \in \Gamma_i \backslash \Gamma} \sum_{k=1}^{\infty} \frac{(\delta_i^{-1} \delta)'(z)^2}{k^2} =$$

$$= -\frac{\pi}{12} \sum_{i=1}^{n} \sum_{\delta \in \Gamma_i \backslash \Gamma} (\delta_i^{-1} \delta)'(z)^2 = -\frac{\pi}{12} \sum_{i=1}^{n} \mathscr{E}_i(z),$$

where

$$\mathscr{E}_i(z) = \sum_{\delta \in \Gamma_i \backslash \Gamma} (\delta_i^{-1} \delta)'(z)^2$$

is the Einsenstein series of weight 4 for the group Γ associated with the cusp z_i, $i = 1, \ldots, n$. Since Eisenstein series are ortogonal to cusp forms with respect to the Petersson scalar product (2) (see [6]) and every $\mu \in \Omega_{-1,1}(\Gamma)$ has a form $\mu = y^2 \bar{\varphi}$, $\varphi \in \Omega_2(\Gamma)$, we see that the integral we are interested in is in fact equal to zero. ∎

THEOREM 2. *For every* $\mu, \nu \in \Omega_{-1,1}(\Gamma)$

$$L_{\bar{\nu}} \left(L_\mu \log \frac{Z'(1)}{\det \operatorname{Im} \tau} \right) = \frac{1}{12\pi} \langle \mu, \nu \rangle -$$

$$- \frac{\pi}{9} \sum_{i=1}^{n} \int_{\Gamma \backslash H} E_i(\cdot, 2) \mu \bar{\nu} \rho,$$

where

$$E_i(z, s) = \sum_{\delta \in \Gamma_i \backslash \Gamma} (\operatorname{Im}(\delta_i^{-1} \delta z))^s, \qquad \operatorname{Re} s > 1,$$

is the Eisenstein-Maass series for the group Γ *associated with the cusp* z_i, $i = 1, \ldots, n$.

Proof. As it follows from Theorem 1,

$$L_{\bar{\nu}}\, L_{\mu} \log Z'(1) = -4 \left.\frac{\partial}{\partial \bar{\epsilon}}\right|_{\epsilon=0} \int_{\Gamma^{\epsilon\nu}\setminus H} R^{\epsilon\nu}\,|_D \cdot \mu^{\epsilon\nu} =$$

$$= -4 \int_{\Gamma\setminus H} (L_{\bar{\nu}}(R\,|_D) \cdot \mu + R\,|_D \cdot L_{\bar{\nu}}\, \mu).$$

The kernel R is regular at the diagonal D in $H \times H$, therefore $L_{\bar{\nu}}(R\,|_D) = (L_{\bar{\nu}} R)|_D$. Recall that

$$R(z, z') = \frac{\partial^2}{\partial z\, \partial z'} (G_1^0(z, z') - Q_1(z, z'),$$

and let us vary the kernels

$$\frac{\partial^2 Q_1}{\partial z\, \partial z'} \quad \text{and} \quad \frac{\partial^2 G_1^0}{\partial z\, \partial z'}$$

separately.

Since

$$\frac{\partial^2 Q_1}{\partial z\, \partial z'}(z, z') = -\frac{1}{4\pi(z - z')^2}$$

a simple computation analogous to that of [4, §4.4] shows that

$$\left.\left(L_{\bar{\nu}} \frac{\partial^2 Q_1}{\partial z\, \partial z'}\right)\right|_D = \frac{1}{48\pi} y^{-2}\, \bar{\nu},$$

and

$$4 \int_{\Gamma\setminus H} \left.\left(L_{\bar{\nu}} \frac{\partial^2 Q_1}{\partial z\, \partial z'}\right)\right|_D \cdot \mu = \frac{1}{12\pi} \langle \mu, \nu \rangle.$$

Further,

$$\frac{\partial^2 G_1^0}{\partial z \partial z'}(z, z') = -\frac{1}{4\pi}\, \Omega(z, z'),$$

where Ω is the Schiffer kernel on $\Gamma \setminus H \times \Gamma \setminus H$ (see formula (8)). From (6), (7) it immediately follows that

$$-4\int_{\Gamma\backslash H}\left(L_{\bar\nu}\frac{\partial^2 G_1^0}{\partial z\,\partial z'}\right)\Bigg|_D\cdot\mu=$$

$$=\int_{\Gamma\backslash H}\left(L_{\bar\nu}\left(\frac{1}{\pi}B(z,z')-\sum_{i,j=1}^{g}(\operatorname{Im}\tau)_{ij}^{-1}\omega_i(z)\omega_j(z')\right)\right)\Bigg|_{z'=z}$$

$$\mu(z)\,dx\,dy=L_{\bar\nu}L_{\mu}\log\det\operatorname{Im}\tau\;-$$

$$-\int_{\Gamma\backslash H}\sum_{i,j=1}^{g}(\operatorname{Im}\tau)_{ij}^{-1}\omega_i\omega_j\,L_{\bar\nu}\mu.$$

In consequence of formula (4) the last integral vanishes and we obtain

$$-4\int_{\Gamma\backslash H}L_{\bar\nu}(R\,|_D)\cdot\mu=L_{\bar\nu}L_{\mu}\log\det\operatorname{Im}\tau\;+\;\frac{1}{12\pi}\langle\mu,\nu\rangle.$$

It remains only to evaluate the integral

$$-4\int_{\Gamma\backslash H}R\,|_D\cdot L_{\bar\nu}\mu$$

(which converges absolutely by Lemma 1). Denote by F the canonical fundamental domain of the group Γ in H such that its cusps are exactly $z_1,\ldots,z_n$. Set $F^Y=\{z\in F\,|\,\operatorname{Im}(\delta_i^{-1}z)\leqslant Y,\,i=1,\ldots\,n\}$ and $l_i=F\cap\{z\in H\,|\,\operatorname{Im}(\delta_i^{-1}z)=Y\}$, $i=1,\ldots,n$. Stokes' formula together with (4) yield

$$\int_{\Gamma\backslash H}R\,|_D\cdot L_{\bar\nu}\,\mu=$$

$$-4\int_F R\,|_D\cdot\frac{\partial}{\partial\bar z}\left(y^2\frac{\partial}{\partial\bar z}(\Delta+2)^{-1}(\mu\bar\nu)\right)\frac{|\,dz\wedge d\bar z\,|}{2}=$$

$$=\frac{2}{\sqrt{-1}}\lim_{Y\to\infty}\int_{F^Y}R\,|_D\cdot\frac{\partial}{\partial\bar z}\left(y^2\frac{\partial f_{\mu\bar\nu}}{\partial\bar z}\right)dz\wedge d\bar z=$$

$$= 2\sqrt{-1} \lim_{Y \to \infty} \oint_{\partial F^Y} R|_D \cdot y^2 \frac{\partial f_{\mu\bar{\nu}}}{\partial \bar{z}} \, dz,$$

where ∂F^Y stands for the boundary of the domain F^Y. Due to Γ-invariance of the integrand, the boundary integral reduces to the sum of integrals over the horocycle arcs l_i, so that

$$\int_{\Gamma \backslash H} R|_D \cdot L_{\bar{\nu}\mu} =$$

$$= 2\sqrt{-1} \lim_{Y \to \infty} \sum_{i=1}^{n} \oint_{l_i} R|_D \cdot y^2 \frac{\partial f_{\mu\bar{\nu}}}{\partial \bar{z}} \, dz =$$

$$= 2\sqrt{-1} \lim_{Y \to \infty} \sum_{i=1}^{n} \oint_{\substack{\mathrm{Re}\, z \in [0,1] \\ \mathrm{Im}\, z = Y}} R|_D (\delta_i z) \cdot (\mathrm{Im}\, \delta_i z)^2 \cdot$$

$$\cdot \frac{\partial f_{\mu\bar{\nu}}}{\partial \bar{z}} (\delta_i z)\, \delta_i'(z)\, dz$$

$$= 2\sqrt{-1} \lim_{Y \to \infty} Y^2 \sum_{i=1}^{n} \oint_{\substack{\mathrm{Re}\, z \in [0,1] \\ \mathrm{Im}\, z = Y}} R|_D (\delta_i z)\, \delta_i'(z)^2 \cdot$$

$$\cdot \frac{\partial}{\partial \bar{z}} (f_{\mu\bar{\nu}}(\delta_i z))\, dz.$$

Recall now that $R|_D$ is a regular automorphic form of weight 4 for the group Γ, whose constant term of Fourier expansion at each cusp z_i is $-\pi/12$ (by [7, Cor. 3.5] it is equal to that of the sum

$$\sum_{\gamma \in \Gamma_i, \gamma \neq 1} \frac{\partial^2}{\partial z \partial z'} Q_1(z, \gamma z')\Big|_{z' = z};$$

see also the computation at the end of the proof of Theorem 1). Substituting the corresponding Fourier series for $R|_D$ and $\partial/\partial\bar{z}\, f_{\mu\bar{\nu}}$ (see Lemma 1) into last formula, we obtain that

$$\int_{\Gamma\backslash H} R|_D \cdot L_{\bar\nu}\mu =$$

$$= -2\sqrt{-1}\lim_{Y\to\infty} Y^2\left(\sum_{i=1}^{n}\left(-\frac{\pi}{12}\right)\cdot\frac{\sqrt{-1}}{2}\left(-\frac{1}{Y^2}\right)c^{(i)}_{\mu\bar\nu} + \right.$$

$$\left. + \underset{Y\to\infty}{o}\left(\frac{1}{Y^2}\right)\right) = \frac{\pi}{12}\sum_{i=1}^{n} c^{(i)}_{\mu\bar\nu} .$$

On the other hand, from the differential equation $\Delta E_i(z, 2) = -2E_i(z, 2)$ and Green's formula we get

$$\int_{\Gamma\backslash H} E_i(\cdot\,, 1)\,\mu\bar\nu\rho =$$

$$= \int_{\Gamma\backslash H}(E_i(z, 2)\,\Delta f_{\mu\bar\nu} - \Delta\,E_i(z, 2)\cdot f_{\mu\bar\nu})\,\frac{dx\,dy}{y^2} =$$

$$= \lim_{Y\to\infty}\oint_{\partial F^Y} E_i(z, 2)\left(\frac{\partial f_{\mu\bar\nu}}{\partial y}\,dx - \frac{\partial f_{\mu\bar\nu}}{\partial x}\,dy\right) -$$

$$- f_{\mu\nu}\left(\frac{\partial}{\partial y}\,E_i(z, 2)\,dx - \frac{\partial}{\partial x}\,E_i(z, 2)\,dy\right).$$

The last integral can easily be evaluated in terms of Fourier coefficients of the functions $f_{\mu\bar\nu}$ and $E_i(z, 2)$. Recall that at the cusp z_j

$$E_i(\delta_j z, 2) = \delta_{ij}\,y^2 + \varphi_{ij}y^{-1} + \underset{y\to\infty}{O}(e^{-2\pi y}),\ i, j = 1, \ldots, n,$$

with some constants φ_{ij} (see, e.g., [7] or [12]), so that

$$\int_{\Gamma\backslash H} E_i(\cdot\,, 2)\,\mu\bar\nu\rho =$$

$$= -\lim_{Y\to\infty}\sum_{j=1}^{n}\left(\delta_{ij}Y^2\cdot\left(-\frac{c^{(j)}_{\mu\bar\nu}}{Y^2}\right) - 2\delta_{ij}Y\cdot\frac{c^{(j)}_{\mu\bar\nu}}{Y}\right) + \underset{Y\to\infty}{o}(1) = 3c^{(i)}_{\mu\bar\nu}$$

and

$$-4\int_{\Gamma\backslash H} R|_D \cdot L_{\bar\nu}\mu = -\frac{\pi}{3}\sum_{i=1}^{n} c_{\mu\bar\nu}^{(i)} =$$

$$= -\frac{\pi}{9}\sum_{i=1}^{n}\int_{\Gamma\backslash H} E(\cdot\,,2)\,\mu\bar\nu\rho,$$

which concludes the proof.

6. REMARK 1. For $X \cong \Gamma\backslash H$ and $\mu, \nu \in \Omega_{-1,1}(\Gamma)$ set

$$\langle\mu,\nu\rangle_{cusp} = \sum_{i=1}^{n}\int_{\Gamma\backslash H} E_i(\cdot\,,2)\,\mu\bar\nu\rho.$$

Since $E_i(z, 2) > 0$ for any $i = 1, \ldots, n$, the Hermitian form $\langle\ ,\ \rangle_{cusp}$ is positive on the linear space $\Omega_{-1,1}(\Gamma) \cong T_{[X]}T_{g,n}$. Moreover, the form $\langle\ ,\ \rangle_{cusp}$ defines a Hermitian metric on the Teichmüller space $T_{g,n}$, $n > 0$, because the Eisenstein-Maass series are real analytic with respect to coordinates on $T_{g,n}$.

As the Weil-Petersson metric is Kähler, by Theorem 2 the metric $\langle\ ,\ \rangle_{cusp}$ on $T_{g,n}$ is also Kähler. Denote by ω_{cusp} its symplectic form; $\omega_{cusp}(\mu,\nu) = \sqrt{-1}/2\ \langle\mu,\nu\rangle_{cusp}$. Then Theorem 2 means that

$$\partial\bar\partial \log \frac{Z'(1)}{\det \operatorname{Im}\tau} = -\frac{\sqrt{-1}}{6\pi}\,\omega_{WP} + \frac{2\sqrt{-1}\pi}{9}\,\omega_{cusp}, \tag{12}$$

where ∂ and $\bar\partial$ are the components of the exterior derivative operator on $T_{g,n}$.

REMARK 2. Since the scalar product $\langle\mu,\nu\rangle_{cusp}$, $\mu, \nu \in \Omega_{-1,1}(\Gamma)$, does not depend on a marking of $X \cong \Gamma\backslash H$, the metric $\langle\ ,\ \rangle_{cusp}$ and its symplectic form ω_{cusp} are defined on the moduli space $\mathcal{U}_{g,n}$ of Riemann surfaces of type (g, n), $n > 0$. In general, the closed $(1, 1)$-forms ω_{WP} and ω_{cusp} on $\mathcal{U}_{g,n}$ represent nontrivial linearly independent cohomology classes $[\omega_{WP}], [\omega_{cusp}] \in H^2(\mathcal{U}_{g,n}; \mathbb{R})$. For instance, when $g \geqslant 3$, $n = 1$, by Theorem 2 we have

$$\frac{1}{12\pi^2}[\omega_{WP}] - \frac{1}{9}[\omega_{cusp}] = pr^*(c_1(\lambda_H)) \neq 0$$

in $H^2(\mathcal{U}_{g,1}; \mathbb{R})$, where λ_H is the Hodge line bundle on $\mathcal{U}_g$ (see [13]), $c_1(\lambda_H) \in H^2(\mathcal{U}_g; \mathbb{Z})$ is its Chern class and pr: $\mathcal{U}_{g,1} \to \mathcal{U}_g$ is the natural projection map ("forgetting of punctures"). Integrals of the forms ω_{WP} and ω_{cusp} over fibers of pr: $\mathcal{U}_{g,1} \to \mathcal{U}_g$ (i.e. compact Riemann surface of genus g) are positive, so both $[\omega_{WP}]$ and $[\omega_{cusp}]$ are not proportional to $\mathrm{pr}^*(c_1(\lambda_H))$. This proves their non-triviality as well as linear independence. Besides, $[\omega_{WP}]$ and $[\omega_{cusp}]$ form a basis of $H^2(\mathcal{U}_{g,1}; \mathbb{R})$, because $\dim H^2(\mathcal{U}_{g,1}; \mathbb{R}) = 2$ (see [14]).

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